SIMPLIFIED SOLUTIONS TO DETERMINE HYDROSTATIC STRESS AND SURFACE BEHAVIOR IN EXTRUSION (DRAWING).

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Abstract
Using Hill's general method of analysis for metal forming processes, methods for finding the distribution of the hydrostatic stress in extrusion (drawing) and mechanical behavior at an interface region are proposed. The approximating velocity field is determined from the upper bound theorem and accounts for the behavior of the real velocity field at the sites where such defects as central burst (about the axis of symmetry) and surface checking (about the interface between the plastic and dead zones) occur. A ductile fracture criterion is applied for central burst control. For the interface region, the velocity field is singular for perfectly/plastic materials, therefore, no ductile fracture criterion can be used for surface checking control. Two concepts are applied at the interface region: (a) rate intensity factor concept and (b) viscous layer concept, to evaluate the severity of deformation gradient.

1. INTRODUCTION
In order to produce a sound, defect-free product, process variables should be optimized so that certain product qualities can be controlled to satisfy particular requirements. The upper bound method is a convenient tool for analysis of metal forming processes. In the case of axisymmetric extrusion (drawing) this technique has been applied by many researchers (see, for example, a review in [1]). Also, in [1] a general kinematically admissible velocity field has been proposed. Predictions based on the upper bound method can be improved if a kinematically admissible velocity field chosen accounts for the behavior of the real velocity field, in addition to the formal requirements of the upper bound theorem. An obvious additional requirement is the behavior of velocity in a vicinity of the axis of symmetry where the shear stress vanishes. Since this stress is calculated from the velocity field and the normality rule, the velocity field can be assumed such that the aforementioned condition is automatically satisfied for any values of parameters involved in the definition of this field. Another additional requirement follows from the known singular behavior of the real velocity field in the vicinity of maximum shear stress surfaces [2]. Singular velocity fields have already been applied in limit analysis and led to improved results as compared with other velocity fields of the same level of complexity [3, 4]. Of course, local improvements in kinematically admissible velocity fields cannot significantly influence the overall load. However, it is believed that such fields may be useful for describing local phenomena such as central bursting at the axis of symmetry and surface checking in extrusion and drawing processes. Even though the upper bound technique has been used for central burst predictions [5], it is clear that it is not an appropriate tool since the hydrostatic stress is not determined from the upper bound theorem for incompressible materials. On the other hand, it is well known that this stress component has a great influence on ductile fracture [6]. A review of other works on the initiation of central burst in extrusion is given in [7] where a criterion for predicting this defect is also proposed. The criterion, as many other criteria for ductile fracture, involves the hydrostatic stress. Therefore, in order to use results of upper bound analyses for fracture predictions it is necessary to develop a procedure for finding this stress. A possible numerical approach has been proposed in [8]. In the present paper, we develop a semi-analytical approach based on the method proposed in [9]. Other applications of this method are given in [10, 11]. Another difficulty with applications of the upper bound theorem to hot extrusion is that simple kinematically admissible velocity fields are usually involve velocity discontinuity surfaces that is impossible in the case of constitutive models suitable for the modeling
of hot metal forming processes. In addition, dead zones cannot exist in the case of constitutive laws with no yield stress [12]. Nevertheless, discontinuous velocity fields are often combined with rate dependent constitutive laws [13, 14]. Since these predictions are satisfactory, one can expect that the strain rate effects are localized in a narrow zone replaced with the velocity discontinuity surface. However, for predictions of local effect, such as surface checking, this zone should be treated separately. A possible approach has been proposed in [2] where a strain rate intensity factor has been introduced by analogy with the stress intensity factor in crack mechanics. The strain rate intensity factor is involved in the coefficient of the singular term in the expression for the equivalent strain rate which approaches infinity on the velocity discontinuity surfaces. In the present paper this factor is calculated for extrusion and used as a fracture criterion. In addition, the analogy to crack mechanics is further developed introducing a layer of viscous material in the vicinity of the velocity discontinuity surface to avoid the singularity in the equivalent strain rate. A similar approach in crack mechanics has been proposed in [15] where a layer of plastic material was introduced to avoid stress singularity in elastic solutions.

2. DISTRIBUTION OF THE HYDROSTATIC STRESS

The virtual work-rate principle for a continuum with no body force and inertia effects can be written in the form:

\[ \int \int \int \int \int \Omega \sigma_{ij} \xi^{ij} dV = \int \int \Omega t_i w' d\Omega \]  

(1)

where \( \sigma_{ij} \) are the components of the stress tensor, \( \xi^{ij} \) are the components of the strain rate tensor, \( t_i \) are the applied surface tractions, \( V \) is the volume of a region of interest and \( \Omega \) is its surface. If equation (1) is valid for all virtual velocity fields \( w' \), the distribution of stress is in equilibrium with \( t' \). The main idea of the method proposed in [9] is to replace \( w' \) with a sufficiently wide class of virtual motions (orthogonalizing motion in the terminology of [9]). In the present paper we assume that an approximating velocity field, \( u_i \), satisfying the incompressibility equation has been found using the upper bound method and seek for a procedure for obtaining a distribution of the hydrostatic stress \( \sigma \) corresponding this velocity field and approximately satisfying the equilibrium equations. Steady axisymmetric flows are considered in a spherical coordinate system \( \rho \theta \phi \) assuming that the \( -\theta \) and \( -\phi \) components of the velocity in the orthogonalizing motion vanish. Using this assumption, a general orthogonalizing motion can be written in the form

\[ w_{\rho} \equiv w_{\rho} , \quad w_{\theta} = 0 , \quad w_{\phi} = 0 \]  

(2)

Here and in what follows the physical components of vectors and tensors are used. In general, \( w \) is an arbitrary function of \( \rho \) and \( \theta \). Its more specific form should satisfy a number of requirements formulated in [9]. It will be chosen later. Since the material is incompressible, the deviatoric components of the stress tensor can be calculated from the velocity field \( u_i \) and a set of constitutive equations, and the hydrostatic stress is an arbitrary function of \( \rho \) and \( \theta \). In the case of the exact solution the hydrostatic stress should be compatible with two equilibrium equations and boundary conditions. To obtain an approximate solution, we mention that the equilibrium equation

\[ \frac{\partial \sigma_{\rho\theta}}{\partial \rho} + \frac{1}{\rho} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{1}{\rho} \left[ (\sigma_{\theta\theta} - \sigma_{\rho\rho}) \cot \theta + 3 \sigma_{\rho\rho} \right] = 0 \]  

(3)

can be solved with respect to \( \sigma \) with no difficulty and its solution will contain one arbitrary function of \( \rho \). This function of \( \rho \) may be found using the method [9]. To apply the method to an
extrusion/drawing process, assume that the surface $\Omega$ consists of four parts defined by the following equations 

$$0 = \theta = 0, \quad 0 = \theta > 0, \quad \rho = \rho_0, \quad \rho = \rho_1 \text{ with } \rho_1 > \rho_0$$

where $\theta = 0$ is the axis of symmetry, $\theta = \theta_0$ is the boundary between the plastic and dead zones, and $\rho = \rho_0$ and $\rho = \rho_1$ are the rigid/plastic boundaries at the exit and entry. At $\theta = \theta_0$ the applied traction $t_\rho = k$ where $k$ is the shear yield stress. At the surfaces $\rho = \rho_0$ and $\rho = \rho_1$ the tractions are not given, but are a part of the solution. The following condition in integral form exists

$$\iiint_{\Omega} \sigma_{ij} n_i w_j d\Omega = 0 \quad \text{(4)}$$

where the surface $\Omega$ is defined either by $\rho = \rho_0$ (extrusion) or $\rho = \rho_1$ (drawing) and $n_i$ are the components of the unit normal vector to this surface. Using (2) the right hand side of (1) for the region under consideration becomes

$$\iiint_{\Omega} t \cdot w' d\Omega = 2\pi \int_0^{\theta} \left( \sigma_{\rho \rho} w^2 \right)_{\rho = \rho_1} \sin \theta d\theta + 2\pi \int_0^{\rho_1} \left( k w \sin \theta \right)_{\theta = 0} \rho d\rho$$

(5)

The left hand side of (1) is

$$\iiint_{\Omega} \left[ \sigma_{\rho \rho} \xi^\rho \xi^\rho + \sigma_{\theta \theta} \xi^\theta \xi^\theta + \sigma_{\rho \rho} \xi^\rho \xi^\rho + 2\sigma_{\rho \rho} \xi^\rho \xi^\rho \right] \rho^2 \sin \theta d\rho d\theta =

= 2\pi \int_0^{\theta} \left( \sigma_{\rho \rho} \frac{\partial w}{\partial \rho} + \sigma_{\theta \theta} \frac{w}{\rho} + \sigma_{\rho \rho} \frac{w}{\rho} + \sigma_{\rho \rho} \frac{\partial w}{\partial \theta} \right) \rho^2 \sin \theta d\rho d\theta$$

(6)

Applying integration by parts to the first and last terms on the right we obtain

$$\int_0^{\theta_0} \rho^2 \sin \theta \sigma_{\rho \rho} \frac{\partial w}{\partial \rho} d\rho d\theta = \int_0^{\theta_0} \left( \rho^2 \sigma_{\rho \rho} w \right)_{\rho = \rho_1} \sin \theta d\theta - \int_0^{\theta_0} \frac{\partial}{\partial \rho} \left( \rho^2 \sigma_{\rho \rho} \right)_{\rho = 0} \rho d\rho \sin \theta d\theta$$

(7)

and

$$\int_0^{\theta_0} \rho^2 \sin \theta \sigma_{\rho \rho} \frac{\partial w}{\partial \theta} d\rho d\theta = \int_0^{\theta_0} \left( \rho \sigma_{\rho \rho} \sin \theta w \right)_{\theta = 0} d\rho - \int_0^{\theta_0} \rho \frac{\partial (\rho \sigma_{\rho \rho} \sin \theta)}{\partial \theta} d\rho$$

(8)

Substituting (5) and (6), with the use of (7) and (8), into (1) gives

$$\int_0^{\theta_0} \int_{\rho_0}^{\rho_1} \rho^2 \sin \theta \frac{\partial \sigma}{\partial \rho} d\rho d\theta + \rho^2 \sin \theta \frac{\partial s_{\rho \rho}}{\partial \rho} + \rho \left( s_{\rho \rho} - s_{\theta \theta} - s_{\rho \rho} \right) \sin \theta + \frac{\partial \left( \rho \sigma_{\rho \rho} \sin \theta \right)}{\partial \theta} \theta = 0$$

(9)

where $\sigma$ is the hydrostatic stress and $s_{\rho \rho}$, $s_{\theta \theta}$, and $s_{\rho \rho}$ are the deviatoric portions of the stress tensor. In the exact solutions the last term in (9) vanishes since the boundary condition $\sigma_{\rho \rho} = k$ at $\theta = \theta_0$ is satisfied. In general, in approximate solutions $\sigma_{\rho \rho} \neq k$ at $\theta = \theta_0$ and, therefore, the last
term in (9) either leads to an approximate boundary condition or becomes a part of an approximate equilibrium equation, like in the case of the slab method. An advantage of using singular solutions found in [2] is that the condition \( \sigma_{\rho\theta} = k \) at \( \theta = \theta_0 \) is satisfied for any kinematically admissible velocity field involving such singular solutions. Therefore, equation (9) simplifies to

\[
\int_0^{\theta_0} \int_0^\rho \sin \theta \frac{\partial \sigma}{\partial \rho} + \sin \theta \left( \frac{\partial (\rho^3 s_{\rho\theta})}{\partial \rho} + \frac{\partial (\rho \sigma_{\rho\theta} \sin \theta)}{\partial \theta} \right) w d\rho d\theta = 0
\]

(10)

where the identity \( s_{\rho\theta} + s_{\theta\theta} + s_{\varphi\varphi} = 0 \) has been taken into account. According to the method [9], the orthogonalizing motion \( w \) involved in (10) should be compatible with the assumed class of \( \sigma \) distribution. In the case under consideration, after solving (3) this distribution will contain one arbitrary function of \( \rho \). Therefore, \( w \) must also contain one arbitrary function of \( \rho \). It is convenient to put

\[
w(\rho) = w_0(\rho)/\rho^2
\]

(11)

Substituting (11) into (10) and integrating with respect to \( \theta \) gives

\[
\int_{\rho_0}^{\rho_1} \left[ \frac{dp}{d\rho} + \frac{1}{\rho^3} \frac{d}{d\rho} (\rho^3 z) + \frac{k \sin \theta_0}{\rho} \right] w_0(\rho) d\rho = 0
\]

(12)

Since \( w_0(\rho) \) is a completely arbitrary function of \( \rho \), it follows from (12) that

\[
\frac{dp}{d\rho} + \frac{1}{\rho^3} \frac{d}{d\rho} (\rho^3 z) + \frac{k \sin \theta_0}{\rho} = 0
\]

(13)

where \( p \) and \( z \) are functions of \( \rho \) defined by

\[
p = \int_0^{\theta_0} \sigma \sin \theta d\theta \quad \text{and} \quad z = \int_0^{\theta_0} s_{\rho\theta} \sin \theta d\theta
\]

(14)

Combining this equation with the solution to (3) and the boundary condition (4) determines the distribution of \( \sigma \) in the plastic zone.

3. RADIAL APPROXIMATING VELOCITY FIELD

A general solenoidal velocity field for axisymmetric extrusion of a rod has been proposed in [1] in the form

\[
u_p = -g(\theta) \begin{pmatrix} g(\theta) \cos \theta + g'(\theta) \sin \theta \end{pmatrix} = \frac{-u(\theta)}{\sin^2 \theta_0 \rho^2}, \quad u_\theta = 0, \quad u_\varphi = 0
\]

(15)

where \( g(\theta) \) is an arbitrary function of \( \theta \) and the equation \( \theta = \theta_0 \) defines either the die surface or the interface between dead and plastic zones. In the latter case the value of \( \theta_0 \) is unknown. The shape of rigid/plastic boundaries at the entry and exit is defined by the following equations [1]
\[ \rho = \frac{g(\theta)}{\sin \theta_0} \quad \text{and} \quad \rho = \frac{g(\theta)}{\sqrt{\lambda} \sin \theta_0} \]  

(16)

respectively. It has been assumed, without the loss of generality, that the entry velocity of the billet is unity and that the radius of the billet is unity. The extrusion ratio is defined by \( \lambda = 1/b^2 \) (\( \lambda > 1 \)) where \( b \) is the final radius of the rod. Using (15) the components of the strain rate tensor and the equivalent strain rate, \( \xi_{eq} = \sqrt{(2/3)\xi_{ij}\xi^{ij}} \), can be obtained. In particular,

\[ \frac{\xi_{\rho\rho}}{\xi_{eq}} = -\frac{1}{2} \frac{\xi_{\rho\rho}}{\xi_{eq}} = -\frac{1}{2} \xi_{\rho\rho} = \frac{1}{\sqrt{1 + h(\theta)}} \quad \text{and} \quad \frac{\xi_{\rho\theta}}{\xi_{eq}} = \frac{\sqrt{3}}{2} \frac{h(\theta)}{\sqrt{1 + h(\theta)}} \]  

(17)

with

\[ h(\theta) = \frac{1}{12} \left[ 3gg'\cos \theta + \left( g'^2 - g^2 + gg'' \right) \sin \theta \right]^2 \quad \frac{g^2 (g \cos \theta + g' \sin \theta)^2}{(18)} \]

The normality rule for a material satisfying Mises' yield criterion gives

\[ s_{ij} = \frac{2}{\sqrt{3}} \xi_{eq} \sqrt{s_{ij}} k \]  

(19)

Substituting (18) into (19) leads to

\[ s_{\rho\rho} = -2s_{\rho\rho} = \frac{2k}{\sqrt{3} \sqrt{1 + h(\theta)}} \quad \text{and} \quad s_{\rho\theta} = \sigma_{\rho\theta} = \frac{k \sqrt{h(\theta)}}{\sqrt{1 + h(\theta)}} \]  

(20)

For a perfectly/plastic material \( k = \text{const} \) and, therefore, \( z \) defined in equation (14) is constant, \( z = kz_0 \), and equation (13) transforms to

\[ \frac{dp}{dp} + k \left( 3z_0 + \sin \theta_0 \right) = 0 \]  

(21)

This equation can be immediately integrated to give

\[ p = -k(3z_0 + \sin \theta_0) \ln(p/p_0) + kp_0 \]  

(22)

where \( p_0 \) is constant. Substituting (20) into (3) and integrating gives

\[ \frac{\sigma}{k} = \frac{1}{\sqrt{3} \sqrt{1 + h(\theta)}} - \frac{3}{k} \int_0^\rho \frac{\sqrt{h(\chi)}}{\sqrt{1 + h(\chi)}} d\chi + \sigma_0(p) \]  

(23)

where \( \chi \) is an auxiliary variable and \( \sigma_0(p) \) is an arbitrary function of \( p \). Using (23) it is possible to find \( p \) introduced in (14) in the form
Combining (22) and (24) results in

\[
\sigma_0 = \left[ p_0 - \frac{z_0}{2} + \frac{3}{\sin \theta_0} \left( \frac{h(\chi)}{1 + h(\chi)} \right) d\chi d\theta - (3z_0 + \sin \theta_0) \ln \left( \frac{\rho}{\rho_0} \right) \right] (1 - \cos \theta_0)^{-1}
\]

(25)
giving \( \sigma_0(\rho) \). In order to use the boundary condition (4), it is necessary to account for the velocity discontinuity surface at the exit. Since the plastic work rate vanishes in the rigid zone and the exit end is stress free, the virtual work-rate principle for a region containing the rigid zone, the velocity discontinuity surface and an infinitesimal volume of the material on the plastic side of the velocity discontinuity surface leads to

\[
\iint_{\Omega} \sigma_0 n_i u_j d\Omega = k \iiint_{\Omega} \left[ u_e \right] d\Omega
\]

(26)

where \( n_i \) are the components of the unit normal vector to the velocity discontinuity surface \( \Omega \) and \( \left[ u_e \right] \) is the jump in the tangential velocity across \( \Omega \). Using (15) and (16) equation (26) can be rewritten in the form

\[
\int_0^{\theta_0} \left( \sigma_{\rho \theta} \frac{g'}{g} - s_{\rho \theta} - \sigma \right) \sin \theta d\theta = \frac{k}{\sin^2 \theta_0} \int_0^{\theta_0} \left[ 1 + \frac{g^2}{g^2} \left( g^2 \cos \theta - u \sin^2 \theta_0 \right)^2 + g^4 \sin^2 \theta \sin \theta \right] d\theta
\]

(27)

Combining (20), (23), (26) and (27) the equation for \( p_0 \) is obtained. Once this equation is solved, the distribution of the hydrostatic stress can be found from (24) and (26).

4. NUMERICAL SOLUTION FOR EXTRUSION THROUGH FLAT DIE

A review of functions \( g(\theta) \) involved in (15) and used in previous studies has been given in [1]. A singular term describing the behavior of the real velocity field in the vicinity of the dead zone has been introduced in [4]. Comparison with other upper bound solutions based on the velocity field (15) has shown that the solution [4] gives a better prediction. However, the velocity field applied in [4] does not account for the behavior of the real velocity field in the vicinity of the axis of symmetry. Here a velocity field satisfying this condition and the condition at \( \theta = \theta_0 \) is introduced assuming that

\[
g(\theta) = 1 + c(\theta_0^2 - \theta^2)^{\nu/2}
\]

(28)

where \( c \) is an arbitrary constant. Using the upper bound theorem and the procedure developed in [1], \( c, \theta_0 \) and the extrusion force \( P \) can be found as functions of \( \lambda \). This numerical solution is illustrated in Fig.1 where the dimensionless extrusion pressure is defined by \( q = P/(\pi R^2 k) \) and \( R_0 \) is the radius of the billet. In Fig. 1, the parameter \( p_0 \) determined from (27) is also shown. Substituting this parameter in (25) and the result in (23) at \( \theta = 0 \) gives the hydrostatic stress at the axis of symmetry. Applying the fracture criterion proposed in [7], it is possible to find that central burst does not occur for the range of parameters under consideration. In order to study surface checking initiation, it is necessary to consider a region in the vicinity of the velocity discontinuity surface \( \theta = \theta_0 \). It follows from (15) and (28) and from [2] in general case that the equivalent strain
rate is infinite at this surface. Therefore, no direct application of ductile fracture criteria is possible. It is similar to the fact that fracture criteria from strength of materials would predict fracture at any crack tip. In [2], the concept of the strain rate intensity factor has been introduced. The process conditions will affect the strength of different fields (the equivalent strain, temperature and others) about the maximum shear stress surface only through the strain rate intensity factor. Therefore, its critical value can serve as a fracture criterion and is conceptually similar to the fracture toughness in crack mechanics. Using this criterion and Fig. 1 the initiation of surface checking can be predicted.

Fig.1. Variation of the extrusion pressure (a), the position of the rigid/plastic interface (b), the strain rate intensity factor (c), and the parameter \( p_0 \) (d) involved in (25) with the extrusion ratio \( \lambda \).

For more accurate predictions it is necessary to consider elastic unloading of a material particle when it leaves the plastic zone. The first step of such analysis is to find the deformation history of the particle in the plastic zone. It follows from the general theory that the state of stress in any particle at the interface between the plastic and dead zones is the shear yield stress with a superimposed hydrostatic stress. The latter is given by (23) at \( \theta = \theta_0 \). The shear yield stress in general depends on the constitutive law chosen. In particular, in the case of perfectly/plastic material it has a constant value. So, for such materials the state of stress has been found. However, as has been mentioned before, the equivalent strain rate approaches infinity at \( \theta = \theta_0 \). This fact has two consequences. First, the state of strain is meaningless. Second, in the vicinity of the surface \( \theta = \theta_0 \) the effect of strain rate cannot be neglected. Our analysis of this zone is based on the idea similar that applied in [15] where a strip of plastic material at the crack tip was introduced in elastic solutions. By analogy, we introduce a thin layer of viscous material in the vicinity of the surface \( \theta = \theta_0 \) in the plastic solution obtained. Assuming that within this layer the equivalent strain rate is constant and that the work rate for the viscous material is equal to the work rate for the plastic material, the thickness of the layer and the equivalent strain rate in the viscous material may be found. By assumption, the thickness of this layer should be small and this should be verified a posteriori. These calculations have been made using the rate dependent model for Al-4Mg alloy
defined in [14]. As a result, the thickness of the viscous layer was found to be very small, order of $10^{-6} - 10^{-7}$ of the initial radius of the billet. This fact confirms that the approach may be applied for introducing rate dependent effects in upper bound solutions based on rigid/perfectly plastic material models.

5. CONCLUSION
An approach to simplified design of extrusion (drawing) processes has been developed. It consists of using the upper bound theorem for finding an approximating velocity field and the method [9] for finding the distribution of the hydrostatic stress corresponding to this velocity field. In addition, the approximating field is chosen such that its behavior accounts for the behavior of the real velocity field at the axis of symmetry and the interface between the plastic and dead zones. Therefore, at these locations more accurate predictions of process variables are obtained. In particular, in the vicinity of the interface the velocity field is singular for perfectly/plastic materials. To treat this region, two concepts are applied: (a) rate intensity factor concept and (b) viscous layer concept, to evaluate the severity of deformation gradient. The rate intensity factor controls many variables of practical interest such as the equivalent strain, the plastic work rate, temperature and others. Therefore, this result may provide a basis for study such physical effects as local heating, recrystallization, and transformations in a narrow layer near the surface.

LITERATURE